Initialization of the Primitive Equations by the Bounded Derivative Method

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(Manuscript received 31 October 1979, in final form 7 March 1980)

ABSTRACT

Large-amplitude high-frequency motions can appear in the solution of a hyperbolic system containing multiple time scales unless the initial conditions are suitably adjusted through a process called *initialization*. We observe that a solution of such a system which varies slowly with respect to time must have a number of time derivatives on the order of the slow time scale. Given a variable which is characteristic of low-frequency motions (e.g., vorticity), we can apply this observation at the initial time to find constraints which determine the rest of the initial data so that the amplitudes of the ensuing high-frequency motions remain small. Boundary conditions of the system must be taken into account in the derivation of the constraints. This procedure is referred to as the *bounded derivative method*.

For a general linear version of the shallow-water equations, we prove that if the initial kth order time derivative is of the order of the slow time scale, then it will remain so for a fixed time interval. For the corresponding constant coefficient system, we compare the present initialization procedure with the normal mode approach. We then apply the new procedure to initialize the nonlinear shallow-water equations including the effect of orography for both the *midlatitude* and *equatorial* beta plane cases. In the midlatitude case, the initialization scheme based on quasi-geostrophic theory can be obtained from the bounded derivative method by certain simplifying assumptions. In the equatorial case, the bounded derivative method provides an effective initialization scheme and new insight into the nature of equatorial flows.

1. Introduction

Typically the hyperbolic systems which describe atmospheric and oceanic motions contain solutions with more than one time scale. For example, the shallow-water equations govern two classes of motion with different time scales. The low-frequency or Rossby-type motions are considered to be meteorologically significant, while the high-frequency or inertia-gravity motions are considered to be noise. Usually analyzed fields of observed data are not suitable as initial conditions for these systems since large-amplitude high-frequency motions can appear during the numerical integration due to an initial imbalance of mass and momentum.

The process of suitably adjusting the initial fields of pressure (or mass) and velocity to control the high-frequency motions of a prediction model is called *initialization* and various methods have been proposed. The earliest initialization scheme adjusted the wind and pressure fields to be in geostrophic equilibrium. Von Hinkelmann (1951) analyzed a linearized version of the shallow-water equations to show that the geostrophic initial condition effectively reduces the amplitude of the rotational part of the noise waves, but the amplitude of the divergent part of the noise waves remains the same order of magnitude as that of the meteorologically significant Rossby motions.

Since the appearance of inertia-gravity motions is associated with the generation of large horizontal divergence, Charney (1955) proposed an initialization scheme obtained from the assumptions that both the horizontal divergence δ and the time rate of change of horizontal divergence δ_t vanish initially. This combination leads to the well-known balance equation. In practice, the assumption of nondivergent initial velocity is too stringent because even Rossby motions are associated with some horizontal divergence expected from their quasi-geostrophic nature and forcings applied to the system. Phillips (1960) later suggested the initial

[1] The National Center for Atmospheric Research is sponsored by the National Science Foundation.

0022-4928/80/071424-13$07.25

specification of horizontal divergence as expected from the quasi-geostrophic formulation.

The initialization procedure based on the quasi-geostrophic formulation has been adopted, with modification, by many atmospheric modelers (e.g., Houghton *et al.*, 1971). The approach is fairly successful, but problems still exist for ultralong waves and for the tropics where the quasi-geostrophic assumption becomes inaccurate. The presence of various constraints imposed on the initial data for application of the quasi-geostrophic theory to data initialization stimulated development of an alternative approach called dynamic initialization. In this approach the prediction equations are integrated forward and backward around the initial time with a small amount of dissipation to filter out the high-frequency motions, thus attaining a balanced state at the initial time (e.g., Miyakoda and Moyer, 1968; Nitta and Hovermale, 1969; Temperton, 1976). Some advantages of the dynamic initialization approach are that the same predictive difference equations are used in both initialization and forecasting and artificial constraints (such as the ellipticity condition of the balance equation) are not required on the initial data. However, the question of achieving a balanced state in the tropics and for very long planetary waves by the dynamic initialization is still open.

Recently, a new approach called nonlinear normal mode initialization has been developed for global primitive-equation models. Dickinson and Williamson (1972) proposed an initialization method of representing the observed data as a series of the normal mode functions of a linearized version of the primitive-equation model. Once the initial data are represented as a normal mode series, the gravity wave mode coefficients may be set to zero. However, the initial elimination of all the gravity wave modes still leads to the generation of small-amplitude gravity waves during the time integration of the primitive equation model due to nonlinear interactions of the remaining modes (Williamson, 1976). The procedure proposed independently by Baer (1977) and Machenhauer (1977) is to retain small-amplitude gravity modes initially to offset the growth of gravity waves resulting from nonlinear interactions of wave components. Baer and Tribbia (1977) discussed higher order versions of the Baer scheme which ensure that the amplitudes of the high-frequency motions are as small as one desires. Tribbia (1979) compared the effectiveness of the first and second order initialization schemes for the shallow-water equations on an equatorial beta plane. Daley (1978) used the constraints of the Machenhauer approach in a variational scheme to adjust both the wind and mass fields for the barotropic case. Both Daley (1979) and Temperton and Williamson (1979) have demonstrated that initialization schemes based on the nonlinear normal mode approach can effectively handle forcing terms and orography in baroclinic models. Using a tangent plane baroclinic model, Leith (1980) has discussed the relationship between the nonlinear normal mode initialization procedure and the initialization procedure based on quasi-geostrophic theory.

Although the nonlinear normal mode initialization procedure is flexible and can handle the effects of forcing such as orography and ultralong waves in the tropics, the approach requires the construction of the normal modes of the prediction equations. While this requirement usually creates no difficulty for a global or hemispherical model, the construction of normal modes for a limited-area model is difficult unless the boundary conditions are periodic or the boundary is a solid wall. Hence the prospect of applying the nonlinear normal mode initialization to limited area primitive equation models is an open question. However, there is an approach called the bounded-derivative method (Kreiss, 1980) which can be used for the initialization of both global and limited-area models.

The purpose of this paper is to discuss the application of the bounded derivative method to initialize the primitive equations as an alternative to the dynamic or nonlinear normal mode initialization schemes. For illustration, we consider the shallow-water equations including the effect of orography as the basic prediction equations, since this system possesses in a simpler context the essential features of the primitive equations. In Section 2, we perform a scale analysis of the set of basic equations to define a relevant system of equations governing motions of interest to us. In Section 3, we introduce the new initialization approach and prove that if the initial kth order time derivative of the dependent variables of a general linear version of the shallow-water equations is of the order of the slow time scale, then it will remain so for a fixed time interval. In Section 4 we examine the bounded derivative method from the view of normal mode solutions of the corresponding constant coefficient system. We apply the method to the full nonlinear equations including the effect of orography for the midlatitude beta plane in Section 5 and to the same problem for the equatorial beta plane in Section 6. Conclusions are presented in Section 7.

2. Scaling of basic equations

In Cartesian coordinates x and y, directed eastward and northward respectively, the shallow-water equations including the effect of orography (e.g., Kasahara, 1966) are expressed by

$$\left.\begin{aligned} &u_t + uu_x + vu_y + gh_x - fv = 0 \\ &v_t + uv_x + vv_y + gh_y + fu = 0 \\ &h_t + (uh)_x + (vh)_y + h_0(u_x + v_y) \\ &\qquad - (uH)_x - (vH)_y = 0 \end{aligned}\right\}, \quad (2.1)$$

where t is time, u and v are the velocity components in the x and y directions, h_0 is the mean height of the homogeneous fluid above sea level, h denotes the deviation of the free surface height from h_0, $H(x,y)$ is the elevation of orography, and g is the constant gravity acceleration. We employ the beta plane so that the Coriolis parameter f varies linearly with y.

To investigate the relative contributions of the various terms in system (2.1), we introduce dimensionless (primed) variables which are simply the original variables divided by their representative magnitudes and thus are of order unity. For the independent variables, we define

$$x' = x/L, \quad y' = y/L, \quad t' = t/T, \qquad (2.2)$$

where L and T are the representative length and time scales, respectively. We also scale the dependent variables by

$$\left.\begin{aligned} u' = u/U, \quad v' = v/U \\ \phi' = h/D, \quad \Phi' = H/H_0 \end{aligned}\right\}, \qquad (2.3)$$

where U is the representative particle speed, D is the representative magnitude of the deviation of the free surface height from its mean, H_0 is the mean elevation of the orography, and ϕ' and Φ' are dimensionless geopotentials corresponding to the free surface height deviation and orography, respectively. We adopt the beta plane convention that $f = 2\Omega(\sin\theta_0 + \cos\theta_0 y/r)$ where Ω is the earth's angular speed, θ_0 is the latitude of the coordinate origin, and r is the earth's radius. We scale f by

$$f' = (2\Omega)^{-1}f = f_0 + R_1\beta y', \qquad (2.4)$$

where $f_0 = \sin\theta_0$, $\beta = \cos\theta_0$, and $R_1 = L/r$.

Since we are interested in meteorologically significant disturbances which are quasi-geostrophic in character (except for a narrow latitudinal band on both sides of the equator) and which move with the speed of order U, we assume that

$$\left.\begin{aligned} D &= 2\Omega UL/g \\ T &= L/U \end{aligned}\right\}. \qquad (2.5)$$

By substituting (2.2)–(2.5) into (2.1), we obtain equations containing only dimensionless variables. For simplicity we write these equations without the prime notation in the form

$$\left.\begin{aligned} &u_t + uu_x + vu_y \\ &\qquad + R_0^{-1}[\phi_x - (f_0 + R_1\beta y)v] = 0 \\ &v_t + uv_x + vv_y \\ &\qquad + R_0^{-1}[\phi_y + (f_0 + R_1\beta y)u] = 0 \\ &\phi_t + (u\phi)_x + (v\phi)_y + R_2(u_x + v_y) \\ &\qquad - R_3[(u\Phi)_x + (v\Phi)_y] = 0 \end{aligned}\right\}, \quad (2.6)$$

where

$$\left.\begin{aligned} R_0 &= U/(2\Omega L) \\ R_2 &= h_0/D \\ R_3 &= H_0/D \end{aligned}\right\}. \qquad (2.7)$$

To evaluate the magnitudes of the dimensionless parameters, we choose

$$\begin{aligned} L &= 10^6 \text{ m}, & h_0 &= 10^4 \text{ m} \\ U &= 10 \text{ m s}^{-1}, & H_0 &= 10^3 \text{ m} \\ 2\Omega &= 10^{-4} \text{ s}^{-1}, & r &= 10^7 \text{ m} \\ g &= 10 \text{ m s}^{-2}, \end{aligned}$$

so that $D = 10^2$ m according to (2.5).

The value of h_0 corresponds to the external mode of atmospheric motion. The particular value of H_0 is selected to conform to the maximum value of mountain height deduced by Phillips (1963) when the motion is to be both quasi-geostrophic and of arbitrary direction with respect to the terrain.

With the above values of scaling, we find that $R_0 = \epsilon$, $R_1 = \epsilon$, $R_2 = \epsilon^{-2}$, and $R_3 = \epsilon^{-1}$, where $\epsilon = O(10^{-1})$. With this notation, system (2.6) can be written in the form

$$\left.\begin{aligned} &u_t + uu_x + vu_y \\ &\qquad + \epsilon^{-1}[\phi_x - (f_0 + \epsilon\beta y)v] = 0 \\ &v_t + uv_x + vv_y \\ &\qquad + \epsilon^{-1}[\phi_y + (f_0 + \epsilon\beta y)u] = 0 \\ &\phi_t + (u\phi)_x + (v\phi)_y + \epsilon^{-2}(\phi_0 - \epsilon\Phi) \\ &\qquad \times (u_x + v_y) - \epsilon^{-1}(u\Phi_x + v\Phi_y) = 0 \end{aligned}\right\}, \quad (2.8)$$

where $\phi_0 = 1$ has been introduced so that any subsequent equation can be transformed into its unscaled equivalent by setting $\epsilon = 1$.

3. Bounded derivative method

Kreiss (1980) has introduced a general initialization procedure, called the bounded derivative method, which is applicable both to pure initial value problems such as those which arise when the entire globe is the domain and to initial-boundary value problems such as those which arise when only a part of the globe is the domain. The procedure is based on the observation that a solution of the scaled equations with a time scale of order unity must have a number of time derivatives of the dependent variables of order unity. The application of this observation at the initial time is used to obtain constraints on the initial data equal in number to that of the large eigenvalues of the system (see Section 4). For the initial-boundary value problem, these constraints are augmented with boundary conditions derived by applying the same observation while taking into account the boundary conditions used for the model. In this section we will prove that if the initial kth order time derivative of a linearized version of system (2.8) is of order unity, then it will remain so for a fixed time interval. For simplicity we will neglect mountain effects, i.e., we will assume $\Phi = 0$.

As the first step in a functional iteration procedure to prove that low-frequency solutions of the nonlinear equations exist, we linearize system (2.8) about the initial data, i.e., we assume the solution can be written in the form

$$\begin{aligned} u &= \bar{u}(x,y) + u'(x,y,t), \\ v &= \bar{v}(x,y) + v'(x,y,t), \\ \phi &= \bar{\phi}(x,y) + \phi'(x,y,t), \end{aligned} \tag{3.1}$$

where the overbar quantities are the initial data and the primed variables are the deviations of the solution from the initial data. In later steps of the iteration, we linearize about the solution obtained from the previous iteration so that the overbar quantities also become a function of t and the following proof becomes more complex. For the details in that case we refer the reader to Kreiss (1980) or Browning (1979).

Substituting (3.1) into (2.8), dropping higher order terms, and removing the prime notation for simplicity, we obtain the linear variable coefficient system

$$\left.\begin{aligned} & u_t + \bar{u}u_x + u\bar{u}_x + \bar{v}u_y + v\bar{u}_y \\ &\qquad + \epsilon^{-1}(\phi_x - fv) + \tilde{u}(x,y) = 0 \\ & v_t + \bar{u}v_x + u\bar{v}_x + \bar{v}v_y + v\bar{v}_y \\ &\qquad + \epsilon^{-1}(\phi_y + fu) + \tilde{v}(x,y) = 0 \\ & \phi_t + \bar{u}\phi_x + u\bar{\phi}_x + \bar{v}\phi_y + v\bar{\phi}_y \\ &\qquad + \epsilon^{-2}(\phi_0 + \epsilon^2\bar{\phi})(u_x + v_y) \\ &\qquad + \phi(\bar{u}_x + \bar{v}_y) + \tilde{\phi}(x,y) = 0 \end{aligned}\right\}, \tag{3.2}$$

where the tilde quantities are spatial forcing terms given by

$$\left.\begin{aligned} \tilde{u}(x,y) &= \bar{u}\bar{u}_x + \bar{v}\bar{u}_y + \epsilon^{-1}(\bar{\phi}_x - f\bar{v}) \\ \tilde{v}(x,y) &= \bar{u}\bar{v}_x + \bar{v}\bar{v}_y + \epsilon^{-1}(\bar{\phi}_y + f\bar{u}) \\ \tilde{\phi}(x,y) &= (\bar{u}\bar{\phi})_x + (\bar{v}\bar{\phi})_y + \epsilon^{-2}\phi_0(\bar{u}_x + \bar{v}_y) \end{aligned}\right\}. \tag{3.3}$$

We assume that the basic state satisfies the relations

$$\left.\begin{aligned} \bar{\phi}_x - f_0\bar{v} &= O(\epsilon) \\ \bar{\phi}_y + f_0\bar{u} &= O(\epsilon) \\ \bar{u}_x + \bar{v}_y &= O(\epsilon^2) \end{aligned}\right\}. \tag{3.4}$$

The above constraints on the basic state ensure that the forcing terms given in (3.3) are all of order unity.

To facilitate the energy proof which follows, we rewrite system (3.2) in symmetric hyperbolic form. To this end, we introduce a new parameter defined by

$$d = (\phi_0 + \epsilon^2\bar{\phi})^{1/2} \tag{3.5}$$

and note that spatial derivatives of d are of order ϵ^2. In operator notation, one symmetric form of (3.2) is given by

$$\left(\frac{\partial}{\partial t} + A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_0\right)Z + F = 0, \tag{3.6}$$

where

$$A_1 = \begin{bmatrix} \bar{u} & 0 & \epsilon^{-3/2}d \\ 0 & \bar{u} & 0 \\ \epsilon^{-3/2}d & 0 & \bar{u} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \bar{v} & 0 & 0 \\ 0 & \bar{v} & \epsilon^{-3/2}d \\ 0 & \epsilon^{-3/2}d & \bar{v} \end{bmatrix},$$

$$A_0 = \begin{bmatrix} \bar{u}_x & -\epsilon^{-1}f + \bar{u}_y & \epsilon^{-3/2}d_x \\ \epsilon^{-1}f + \bar{v}_x & \bar{v}_y & \epsilon^{-3/2}d_y \\ \epsilon^{1/2}d^{-1}\bar{\phi}_x & \epsilon^{1/2}d^{-1}\bar{\phi}_y & d^{-1}(d_x\bar{u} + d_y\bar{v}) + (\bar{u}_x + \bar{v}_y) \end{bmatrix},$$

$$Z = \begin{bmatrix} u \\ v \\ \epsilon^{1/2}d^{-1}\phi \end{bmatrix}, \quad F = \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \epsilon^{1/2}d^{-1}\tilde{\phi} \end{bmatrix}.$$

By differentiating (3.6) with respect to time, we have

$$\left(\frac{\partial}{\partial t} + A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} + A_0\right)\dot{Z} = 0, \quad (3.7)$$

where the dot over Z denotes the time derivative.

Multiplying (3.7) by $\dot{Z}^{\mathrm{T}}$ (the transpose of $\dot{Z}$), we obtain the scalar equation

$$|\dot{Z}|_t^2 + \frac{\partial}{\partial x}(\dot{Z}^{\mathrm{T}}A_1\dot{Z}) + \frac{\partial}{\partial y}(\dot{Z}^{\mathrm{T}}A_2\dot{Z}) + 2\dot{Z}^{\mathrm{T}}E\dot{Z} = 0, \quad (3.8)$$

where $|Z|^2 = Z^{\mathrm{T}}Z$ and

$$2E = 2A_0' - \partial A_1/\partial x - \partial A_2/\partial y.$$

The term A_0' is the same A_0 as in (3.6) except that the Coriolis terms are absent. The disappearance of the Coriolis terms in A_0' is due to the fact that the Coriolis terms appear in antisymmetric form in A_0 so that they do not contribute to the energy estimate. This is an important point because *the terms of E are all of order unity and, therefore, the norm of the operator E is of order unity.*

We integrate (3.8) over the rectangle $R = \{(x,y): x_1 \leq x \leq x_2,\ y_1 \leq y \leq y_2\}$ and after application of the divergence theorem, we have

$$\frac{\partial}{\partial t}\int_R |\dot{Z}|^2 dxdy + \int_{y_1}^{y_2} [\dot{Z}^{\mathrm{T}}A_1\dot{Z}]_{x_1}^{x_2} dy + \int_{x_1}^{x_2} [\dot{Z}^{\mathrm{T}}A_2\dot{Z}]_{y_1}^{y_2} dx + 2\int_R \dot{Z}^{\mathrm{T}}E\dot{Z}dxdy = 0.$$

For periodic or solid wall boundary conditions, the middle terms vanish. For open boundaries, the argument is more complex and we refer the reader to Kreiss (1980). In the first two cases, we can apply the Cauchy-Schwarz inequality to obtain

$$\partial/\partial t \int_R |\dot{Z}|^2 dxdy \leq 2 \max_R |E| \int_R |\dot{Z}|^2 dxdy$$

and this is solved in the form

$$\left[\int\int_R |\dot{Z}(x,y,t)|^2 dxdy\right]^{1/2} \leq \exp[\max_R |E|t]\left[\int\int_R |\dot{Z}(x,y,0)|^2 dxdy\right]^{1/2}. \quad (3.9)$$

Thus, the first-order time derivative $\dot{Z}$ is of order unity for a fixed interval of time if $\dot{Z}$ is initially of order unity. Note that the inequality (3.9) is similar to that appearing in connection with proofs of the well-posedness of physical problems. The reader may be concerned about the exponential growth rate of (3.9). However, this is a theoretical upper bound and the growth rate is much smaller in practice.

The proof above also holds if the dot represents any order time derivative of Z so that if any order time derivative of Z is initially of order unity then it will remain so for a fixed interval of time. Clearly, the larger the number of time derivatives that are of order unity, the smoother the solution will be in time. An analogous proof also applies in the case of the equatorial beta plane system of Section 6.

To apply the bounded derivative method to the first-order time derivatives of system (3.2) at $t = 0$, we must have

$$\left.\begin{aligned} \phi_x - f_0 v &= O(\epsilon) \\ \phi_y + f_0 u &= O(\epsilon) \\ u_x + v_y &= O(\epsilon^2) \end{aligned}\right\}, \quad (3.10)$$

and these are analogous to (3.4) for the basic state. By cross-differentiating the first two equations of (3.10), we obtain the equivalent system

$$\nabla^2\phi - f_0\zeta = O(\epsilon), \quad (3.11)$$

$$\delta = O(\epsilon^2), \quad (3.12)$$

where $\zeta = -u_y + v_x$ and $\delta = u_x + v_y$.

For the case of the midlatitude beta plane in which $f_0 = O(1)$, Eq. (3.11) can be used to determine ϕ to order ϵ given the vorticity ζ or to determine ζ to order ϵ given the geopotential ϕ. The divergence δ must be specified to order ϵ^2 by (3.12). Then u and v can be computed uniquely from ζ and δ by using the Helmholtz theorem as discussed later.

For the case of the equatorial beta plane in which $f_0 = 0$, the first two equations of (3.10) imply that

$$\phi = O(\epsilon). \quad (3.13)$$

By rescaling the third equation of (3.2) using the above relationship, we find a more stringent constraint on the order of the divergence δ, i.e.,

$$\delta = O(\epsilon^3). \quad (3.14)$$

For both cases, more refined initial constraints on ϕ and δ can be obtained by requiring that the second-order time derivatives be of order unity at $t = 0$. Since this process will be carried out for the full nonlinear system (2.8) in Sections 5 and 6, we will not repeat the essentially analogous computations here.

4. Local normal modes of linearized equations

Solutions of the linearized equations (3.6) for the pure initial value problem can be obtained by the superposition of the normal mode solutions with their coefficients determined by the initial conditions. In this case, we can demonstrate the consequences of the initialization procedure obtained from the bounded derivative method. To simplify the normal mode analysis, we only consider the *local* normal modes of the linear equations (3.6) without the forcing term F for the case of the midlatitude beta plane. The local properties of the system can be derived by freezing the coefficients to be their values at a particular point (x,y) in space.

We can Fourier transform the resulting constant coefficient equation as

$$\left(\frac{\partial}{\partial t} + A_1 i\xi_1 + A_2 i\xi_2 + A_0\right)\hat{Z} = 0, \qquad (4.1)$$

where ξ_1 and ξ_2 are the wave numbers in the x and y directions, respectively, and $\hat{Z}$ is the transform of Z.

Eq. (4.1) is a linear constant coefficient system of ordinary differential equations with parameters ξ_1 and ξ_2. Let $A = |\xi|^{-1}(A_1 i\xi_1 + A_2 i\xi_2 + A_0)$, where $|\xi|^2 = (\xi_1^2 + \xi_2^2)$ denotes the usual Euclidean norm. Then A is given by

$$|\xi|A = \begin{bmatrix} is + \bar{u}_x & -\epsilon^{-1}f + \bar{u}_y & \epsilon^{-3/2}di\xi_1 + \epsilon^{-3/2}d_x \\ \epsilon^{-1}f + \bar{v}_x & is + \bar{v}_y & \epsilon^{-3/2}di\xi_2 + \epsilon^{-3/2}d_y \\ \epsilon^{-3/2}di\xi_1 + \epsilon^{1/2}d^{-1}\bar{\phi}_x & \epsilon^{-3/2}di\xi_2 + \epsilon^{1/2}d^{-1}\bar{\phi}_y & is + d^{-1}(d_x\bar{u} + d_y\bar{v}) + \bar{\delta} \end{bmatrix}, \qquad (4.2)$$

where $\bar{\delta} = \bar{u}_x + \bar{v}_y$ and $s = \bar{u}\xi_1 + \bar{v}\xi_2$. In terms of A, we write (4.1) as

$$\hat{Z}_t + |\xi|A\hat{Z} = 0. \qquad (4.3)$$

The solution of (4.3) can be written explicitly in terms of the eigenvalues and eigenvectors of A. Toward this goal we compute the eigenvalues of the matrix

$$\mathbf{A} = |\xi|^{-1}\begin{bmatrix} 0 & -\epsilon^{-1}f_0 & \epsilon^{-3/2}di\xi_1 \\ \epsilon^{-1}f_0 & 0 & \epsilon^{-3/2}di\xi_2 \\ \epsilon^{-3/2}di\xi_1 & \epsilon^{-3/2}di\xi_2 & 0 \end{bmatrix}.$$

This matrix is obtained from (4.2) by keeping only the large terms. The eigenvalues $\boldsymbol{\lambda}$ of $\mathbf{A}$ are solutions of the equation $\det(\mathbf{A} - \boldsymbol{\lambda}\mathbf{I}) = 0$. Thus, the eigenvalues $\boldsymbol{\lambda}$ satisfy the cubic polynomial equation

$$\boldsymbol{\lambda}(\boldsymbol{\lambda}^2 + \epsilon^{-3}d^2 + \epsilon^{-2}|\xi|^{-2}f_0^2) = 0$$

and are given by

$$\boldsymbol{\lambda}_1 = 0,$$

$$\boldsymbol{\lambda}_2 = i\epsilon^{-3/2}(d^2 + \epsilon|\xi|^{-2}f_0^2)^{1/2},$$

$$\boldsymbol{\lambda}_3 = -\boldsymbol{\lambda}_2.$$

Remembering that the matrix $\mathbf{A}$ comprises the large terms of A and the definition of d given in (3.5), we find that the eigenvalues λ of A are

$$\lambda_1 = \boldsymbol{\lambda}_1 + O(1) = O(1),$$

$$\lambda_2 = \boldsymbol{\lambda}_2 + O(1) = O(\epsilon^{-3/2}),$$

$$\lambda_3 = \boldsymbol{\lambda}_3 + O(1) = O(\epsilon^{-3/2}).$$

The eigenvalue λ_1 of order unity corresponds to the Rossby motions. The other two eigenvalues λ_2 and λ_3 of order $\epsilon^{-3/2}$ correspond to the inertia-gravity motions.

The eigenvectors $\hat{Z}_j$, $j = 1, 2$, or 3, of A are solutions of the system of equations $|\xi|(A - \lambda_j I)\hat{Z}_j = 0$, where we can choose $\hat{u}_j$, $\hat{v}_j$, and $\hat{\phi}_j$ so that $|\hat{Z}_j| = 1$. Collecting terms of order unity, this system of equations can be rewritten as

$$\begin{aligned} |\xi|^{-1}\epsilon^{-1}(i\xi_1\hat{\phi} - f_0\hat{v}) &= O(1) + \lambda_j\hat{u}, \\ |\xi|^{-1}\epsilon^{-1}(i\xi_2\hat{\phi} + f_0\hat{u}) &= O(1) + \lambda_j\hat{v}, \\ |\xi|^{-1}\epsilon^{-2}d^2(i\xi_1\hat{u} + i\xi_2\hat{v}) &= O(1) + \lambda_j\hat{\phi}, \end{aligned} \qquad (4.4)$$

where the subscript j for $\hat{u}$, $\hat{v}$ and $\hat{\phi}$ has been dropped.

For the Rossby mode ($j = 1$), $\lambda_1 = O(1)$ so that all the terms on the right-hand side of (4.4) are of order unity. Then the Rossby mode yields

$$\left.\begin{aligned} |\xi|^{-1}(i\xi_1\hat{\phi} - f_0\hat{v}) &= O(\epsilon) \\ |\xi|^{-1}(i\xi_2\hat{\phi} + f_0\hat{u}) &= O(\epsilon) \\ |\xi|^{-1}(i\xi_1\hat{u} + i\xi_2\hat{v}) &= O(\epsilon^2) \end{aligned}\right\}. \qquad (4.5)$$

As is well known, this means that the Rossby mode is quasi-geostrophic and quasi-nondivergent. These relationships imply those of (3.10) which were derived by requiring that the first-order time derivatives be of order unity.

For the inertia-gravity modes ($j = 2, 3$), $\lambda_j = O(\epsilon^{-3/2})$ so that the right-hand sides of (4.4) are of order $\epsilon^{-3/2}$. For these modes, we have

$$\left.\begin{aligned} |\xi|^{-1}(i\xi_1\hat{\phi} - f_0\hat{v}) &= O(\epsilon^{-1/2}) \\ |\xi|^{-1}(i\xi_2\hat{\phi} + f_0\hat{u}) &= O(\epsilon^{-1/2}) \\ |\xi|^{-1}(i\xi_1\hat{u} + i\xi_2\hat{v}) &= O(\epsilon^{1/2}) \end{aligned}\right\}. \qquad (4.6)$$

The general solution of the constant coefficient problem (4.3) can be written as

$$\hat{Z} = \sum_{j=1}^{3} c_j \exp[-|\xi|\lambda_j t]\hat{Z}_j,$$

where the c_j are determined by the initial data. To maintain the quasi-geostrophic character of motion, $\hat{u}$, $\hat{v}$ and $\hat{\phi}$ must satisfy the relationships (4.5) for all time including time $t = 0$. However, for arbitrary initial data, c_2 and c_3 can be of order unity so that in general the relations (4.5) will not be satisfied. From (4.6) we see that the relations (4.5) will hold for all time if and only if c_2 and c_3 are of order $\epsilon^{3/2}$. The bounded derivative method is just the natural extension of this observation, i.e., the kth order time derivative of $\hat{Z}$ is given by

$$\frac{d^k\hat{Z}}{dt^k} = \sum_{j=1}^{3} c_j(-|\xi|\lambda_j)^k \exp[-|\xi|\lambda_j t]\hat{Z}_j$$

and is bounded independently of ϵ if and only if $c_2 = O(\epsilon^{3k/2})$ and $c_3 = O(\epsilon^{3k/2})$. We see that as k approaches infinity, c_2 and c_3 approach zero. In other words, as the order of the time derivative which is bounded independently of ϵ becomes higher, the amplitudes of the inertia-gravity modes become smaller, while the Rossby mode is unaffected because $c_1 = O(1)$.

Based on the normal mode analysis of a system of perturbation equations arising from linearization of the shallow-water equations around a simple basic state, von Hinkelmann (1969) demonstrated that the initial constraints $\partial^k\delta/\partial t^k = 0$ and $\partial^{k+1}\delta/\partial t^{k+1} = 0$ on divergence δ at $t = 0$ will eliminate the inertia-gravity modes from the solution as k approaches infinity. Although bounding the time derivatives of a particular variable will work in specific cases, in general, such a procedure does not ensure that the kth order time derivative of the vector variable Z is of order unity.

5. Initialization of the nonlinear system for the case of the midlatitude beta plane

In this section, we apply the bounded derivative method to the nonlinear system (2.8) for the case of the midlatitude beta plane. For this case, it has been traditional to specify the geopotential and then to attempt to find the corresponding balanced wind components. However, this procedure does not work in the equatorial region where f approaches zero. Also, for the synoptic scale of motions, the mass field tends to adjust to the wind field as shown by the adjustment theory (e.g., Blumen, 1972). It is, then, more desirable to calculate the geopotential field from the wind information rather than vice versa.[2] Thus we will assume that either the streamfunction ψ or the rotational components of the wind u^0 and v^0 are given as initial data.

The first-order time derivative of (2.8) is of order unity if and only if

$$\begin{bmatrix} 0 & -f_0 & \dfrac{\partial}{\partial x} \\ f_0 & 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ \phi \end{bmatrix} = \epsilon \begin{bmatrix} a(x,y,t) \\ b(x,y,t) \\ s^{-1}[u\Phi_x + v\Phi_y + \epsilon c(x,y,t)] \end{bmatrix}; \quad (5.1)$$

where $s(x,y) = \phi_0 - \epsilon\Phi$ and we assume that a, b and c are smooth, i.e., they and their spatial derivatives are of order unity. We place the orographic terms on the right hand side so that the following argument also applies when those terms are absent. In doing so, we are considering them as a perturbation of the operator on the left-hand side since they are an order of magnitude smaller than the terms on the left-hand side.

The operator on the left-hand side of (5.1) is a singular operator and so (5.1) has a solution if and only if

$$-a_y + b_x = f_0 s^{-1}(u\Phi_x + v\Phi_y + \epsilon c). \quad (5.2)$$

At the first stage of the initialization, the simplest way to ensure that (5.1) has a solution at $t = 0$ is to neglect terms of order ϵ in (5.1), i.e., we will assume that the right-hand side of (5.1) is identically zero. Since (5.1) contains only two independent equations, we can choose one quantity arbitrarily and for the reasons stated above, we choose that quantity to be the streamfunction ψ. Then the third equation of (5.1) is automatically satisfied and by defining $\phi = f_0\psi$, we see that (5.1) is satisfied at $t = 0$.

The assumption that the right-hand side of (5.1) is zero is rather crude, as it can be of the form shown provided (5.2) is satisfied. Although we could choose a, b and c arbitrarily so that (5.2) is satisfied, it is better to find additional equations for them by requiring the second-order time derivatives to be of order unity. To compute the second-order time derivatives, we rewrite (2.8) using (5.1) in the form

$$\left.\begin{aligned} u_t + uu_x + vu_y - \beta yv + a &= 0 \\ v_t + uv_x + vv_y + \beta yu + b &= 0 \\ \phi_t + u\phi_x + v\phi_y + \phi(u_x + v_y) + c &= 0 \end{aligned}\right\} . \quad (5.3)$$

Since we have assumed that a, b and c are smooth functions of x and y and we also assume that the initial data are smooth, the first-order time derivatives of u, v and ϕ and the spatial derivatives of u_t, v_t and ϕ_t are of the order of unity. Therefore, u_{tt}, v_{tt} and ϕ_{tt} are of the order of unity if and only

[2] In the middle to higher latitudes where the geopotential field is considered reliable, it is possible to devise an initialization scheme giving only the geopotential as input data. Such a scheme is presented in the Appendix. In a variational version (e.g., Daley, 1978), the geopotential and/or the wind could be varied to minimize the differences between balanced data and observations.

if a_t, b_t and c_t are of the order of unity. Utilizing (5.1) and (5.3) we have

$$\begin{aligned} -\epsilon a_t &= -(\phi_{xt} - f_0 v_t) \\ &= [\epsilon(au + bv) + \phi\delta]_x + c_x \\ &\qquad - f_0(uv_x + vv_y + \beta yu + b), \end{aligned} \tag{5.4}$$

$$\begin{aligned} -\epsilon b_t &= -(\phi_{yt} + f_0 u_t) \\ &= [\epsilon(au + bv) + \phi\delta]_y + c_y \\ &\qquad + f_0(uu_x + vu_y - \beta yv + a), \end{aligned} \tag{5.5}$$

$$\begin{aligned} -\epsilon^2 c_t &= -[s\delta_t - \epsilon(u_t\Phi_x + v_t\Phi_y)] \\ &= s[a_x + b_y - \beta y\zeta + \beta u - 2(u_x v_y - u_y v_x) \\ &\quad + u\delta_x + v\delta_y + \delta^2] \\ &\quad - \epsilon[(uu_x + vu_y - \beta yv + a)\Phi_x \\ &\qquad + (uv_x + vv_y + \beta yu + b)\Phi_y]. \end{aligned} \tag{5.6}$$

We can write (5.4)–(5.6) in operator notation as

$$\begin{bmatrix} 0 & -f_0 & \dfrac{\partial}{\partial x} \\ f_0 & 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} F^{(1)} \\ F^{(2)} \\ F^{(3)} \end{bmatrix}, \tag{5.7}$$

where

$$\begin{aligned} -F^{(1)} &= [\epsilon(au + bv) + \phi\delta]_x \\ &\qquad - f_0(uv_x + vv_y + \beta yu) + \epsilon a_t, \end{aligned} \tag{5.8}$$

$$\begin{aligned} -F^{(2)} &= [\epsilon(au + bv) + \phi\delta]_y \\ &\qquad + f_0(uu_x + vu_y - \beta yv) + \epsilon b_t, \end{aligned} \tag{5.9}$$

$$\begin{aligned} -F^{(3)} &= -\beta y\zeta + \beta u - 2(u_x v_y - u_y v_x) + u\delta_x \\ &\quad + v\delta_y + \delta^2 + \epsilon^2 s^{-1} c_t \\ &\quad - \epsilon s^{-1}[(uu_x + vu_y - \beta yv + a)\Phi_x \\ &\qquad + (uv_x + vv_y + \beta yu + b)\Phi_y]. \end{aligned} \tag{5.10}$$

The operator on the left-hand side of (5.7) is the same as that on the left-hand side of (5.1). Thus (5.7) has a solution if and only if the right-hand side satisfies the constraint

$$-F_y^{(1)} + F_x^{(2)} = f_0 F^{(3)} \tag{5.11}$$

and we note that this constraint is satisfied except for terms of order ϵ. Eq. (5.7) constitutes only two independent equations for the three unknowns a, b, and c. However, we must also satisfy (5.2) which combined with (5.7) determines a, b and c uniquely. Once a, b and c are determined, the divergence δ can be calculated from the third equation of (5.1). The total wind components u and v can then be calculated from the Helmholtz relations

$$\left.\begin{aligned} \nabla^2 u &= \delta_x - \zeta_y \\ \nabla^2 v &= \delta_y + \zeta_x \end{aligned}\right\}. \tag{5.12}$$

Then the geopotential can be determined from either the first or second equations of (5.1). Because the divergent part of the wind is present on the right-hand side of (5.2) and (5.7), an iteration procedure can be used to obtain their solution starting with the initial specification of u and v by their rotational component. Boundary conditions for these equations are derived from the boundary conditions for system (2.8). We will show how this is done for the channel model in Section 6.

Eq. (5.7) must be satisfied by any initial data which is to give rise to a slow time scale solution. However, in order to throw light on the relationship between the bounded derivative approach and the quasi-geostrophic formulation, *we will simplify this general procedure by determining* a, b *and* c *only to order unity, i.e., we will drop terms of order* ϵ.

Dropping terms of order ϵ in (5.2) and then using the first two equations of (5.1) to replace a and b, we obtain an equation for the divergence of the form

$$\delta = \epsilon\phi_0^{-1}(u^0\Phi_x + v^0\Phi_y), \tag{5.13}$$

which states that the divergence is at most of order ϵ. For the external mode ($h_0 = 10^4$ m), the condition $c = 0$ which leads to (5.13) seems sufficient to control inertia-gravity motions. This is no doubt due to the fact that the velocity components are computed with only an error of order ϵ^2. However, for an internal mode such as $h_0 = 10^3$ m, the terms on the right-hand side of (5.4) or (5.5) become of the same order as the orographic terms of (5.13) so that their contribution becomes accordingly significant. Then we can not assume $c = 0$ and must compute c from (5.7). If we differentiate the first and second equations of (5.7) with respect to x and y, respectively, and add the resulting equations, we obtain an equation similar to the divergence equation obtained by Phillips (1960) based on quasi-geostrophic formulations.

Dropping terms of order ϵ in the third equation of (5.7) and then using the first two equations of (5.1) to replace a and b, we have

$$\nabla^2\phi - f\zeta + \epsilon\beta u^0 = 2\epsilon(u_x{}^0 v_y{}^0 - u_y{}^0 v_x{}^0). \tag{5.14}$$

Eq. (5.14) is the so-called balance equation as derived by Charney (1955). Note that to this order of the initialization procedure, the effects of horizontal divergence and orography do not appear explicitly in determining the initial geopotential field. This is consistent with the quasi-geostrophic theory. However, this is not the case for the equatorial beta-plane initialization scheme of the same order accuracy as we shall see in the next section.

6. Initialization of the nonlinear system for the case of the equatorial beta plane

In this section we use the bounded derivative method to obtain an initialization scheme for the equatorial beta-plane system, i.e., we assume $f_0 = 0$. We shall demonstrate that the bounded derivative method not only provides an effective initialization scheme, but also provides insight into some special problems which occur in the tropics.

Near the equator the Coriolis term f is of the order ϵ so that for this case the first two constraints of (5.1) can be written $\phi_x = O(\epsilon)$ and $\phi_y = O(\epsilon)$. These relations imply that in the equatorial region ϕ must be of order ϵ. Of course, this property of the geopotential has been known observationally for a long time, i.e., the geopotential deviation is much smaller in the equatorial region. For the equatorial beta plane case, we use this property to rescale the equations by assuming $D = 10$ m, while leaving the remaining scaling parameters the same for the system (2.6). We find that for the equatorial region the relevant system is

$$u_t + uu_x + vu_y + \phi_x - \beta yv = 0, \tag{6.1}$$

$$v_t + uv_x + vv_y + \phi_y + \beta yu = 0, \tag{6.2}$$

$$\phi_t + (u\phi)_x + (v\phi)_y + \epsilon^{-3}s(u_x + v_y) - \epsilon^{-2}(u\Phi_x + v\Phi_y) = 0, \tag{6.3}$$

where $s = \phi_0 - \epsilon\Phi$.

For this system, the first-order time derivative is of order unity if and only if

$$s\delta - \epsilon(u\Phi_x + v\Phi_y) = \epsilon^3 c, \tag{6.4}$$

where, as before, we assume c to be smooth. For this case we have obtained only one constraint from bounding the first-order time derivative. This is typical of systems in which the large coefficients only appear in one equation. As we need two constraints for proper initialization (since there are two large eigenvalues), in this case we are automatically forced to go to the second-order time derivative to obtain a second constraint.

To compute the second-order time derivative in this case, we need only rewrite (6.3) in the form

$$\phi_t + (u\phi)_x + (v\phi)_y + c = 0. \tag{6.5}$$

Since we have assumed that the initial data and c are smooth functions of x and y, the first-order time derivatives of u, v and ϕ and the space derivatives of u_t, v_t and ϕ_t are of the order unity. Therefore, u_{tt}, v_{tt} and ϕ_{tt} are of order unity if and only if c_t is of order unity. From (6.4) we have

$$\begin{aligned} -\epsilon^3 c_t &= -[s\delta_t - \epsilon(u_t\Phi_x + v_t\Phi_y)] \\ &= s[\nabla^2\phi - \beta y\zeta + \beta u - 2(u_x v_y - u_y v_x) \\ &\quad + u\delta_x + v\delta_y + \delta^2] \\ &\quad - \epsilon[(uu_x + vu_y + \phi_x - \beta yv)\Phi_x \\ &\qquad + (uv_x + vv_y + \phi_y + \beta yu)\Phi_y]. \end{aligned} \tag{6.6}$$

Eqs. (6.4) and (6.6) constitute a system of two initialization constraints and are similar to the constraints (5.13) and (5.14) of the previous section. However, we need to emphasize some important differences between them. We are not allowed to replace u and v in (6.4) by u^0 and v^0 as we did in (5.5) as this would result in an error of order ϵ^2 on the left-hand side of (6.4) so that the first-order time derivative of ϕ would not be ensured to be of order unity. This more restrictive constraint also requires increased accuracy in the numerical model so that the truncation error in the computation of the divergence δ is less than order ϵ^3. Also, if we want to determine ϕ up to order ϵ as in the previous section, then we are not allowed to replace u and v in the Jacobian term of (6.6) by u^0 and v^0 and cannot ignore the terms involving the advection of the divergence δ.

The easiest way to satisfy these constraints is to neglect terms of order ϵ^3. Then given the vorticity ζ, the divergence δ and the geopotential ϕ can be determined from (6.4) and (6.6). Some iteration is necessary to solve this simultaneous system of constraints although it involves only a few steps since a good initial guess is obtained by replacing u and v in (6.4) and (6.6) by u^0 and v^0. At each step of the iteration, u and v can be obtained from ζ and δ from the Helmholtz relations (5.12).

As we shall see from the numerical results, this scheme almost completely controls the inertia-gravity motions. However, we have assumed $c = 0$ in spite of the fact that c is allowed to be of order unity and is then as important as the remaining terms of (6.5). If a better initialization scheme is desired, then it is possible to constrain the third-order time derivative yielding an elliptic equation for c which combined with (6.6) would provide us with a new initialization system. Once c is determined to order unity, we can use the definition of c in (6.4) to compute the corresponding divergence δ. However, we must stress that the initialization scheme and the numerical model must have sufficient resolution to ensure that these refinements are not lost in the truncation errors of either. If this is not the case, then we must apply the analogous principle to the finite-difference equations without recourse to the continuous equations or, as Kreiss (1979) has suggested, take the truncation errors of the model into account in the initialization scheme.

To test the initialization scheme for (6.1)–(6.3), we consider the region $R = \{(x,y): 0 \le x \le X, -Y \le y \le Y\}$. We assume periodic boundary conditions in the x direction and solid walls at $y = -Y$ and $y = Y$ where the normal velocity v must vanish.

For the numerical approximation, we choose the spatial increments $\Delta x = X/I$ and $\Delta y = 2Y/J$, where I and J are natural numbers. The temporal increment Δt is determined from the stability requirement. We define the finite-difference grid

$$G = \{(x_i,y_j,t_n): x_i = (i-1)\Delta x, \quad y_j = -Y + (j-1)\Delta y,\ t_n = n\Delta t\}$$

with

$$1 \le i \le I, \quad 1 \le j \le J+1, \quad \text{and} \quad -1 \le n \le N$$

and employ the standard grid function notation $u_{i,j}^n \approx u(x_i,y_j,t_n)$. We assume that missing subscripts or superscripts are the nominal values i, j or n as appropriate. By defining the difference operators

$$D_t u = \frac{u^{n+1} - u^{n-1}}{2\Delta t},$$

$$D_x u = \frac{u_{i+1} - u_{i-1}}{2\Delta x},$$

$$D_y u = \frac{u_{j+1} - u_{j-1}}{2\Delta y},$$

we can approximate the unscaled version of system (6.1)–(6.3) by

$$\left.\begin{aligned} &D_t u + uD_x u + vD_y u + D_x\phi - \beta y v = 0 \\ &D_t v + uD_x v + vD_y v + D_y\phi + \beta y u = 0 \\ &D_t\phi + u(D_x\phi - \Phi_x) + v(D_y\phi - \Phi_y) \\ &\qquad + (\phi_0 + \phi - \Phi)(D_x u + D_y v) = 0 \end{aligned}\right\}, \tag{6.7}$$

where Φ, Φ_x, and Φ_y are known functions. For given n, the finite-difference equations (6.7) can be applied at each spatial grid point of G except on the boundaries at $y = -Y$ and $y = Y$. On these boundaries, the equation for u can be used as before since the term $vD_y u$ is zero there. The equation for v is not required as $v = 0$ on the boundaries. In the equation for ϕ, the term $v(D_y\phi - \Phi_y)$ is zero and we approximate the term v_y by the appropriate one-sided, second-order difference formula. When $i = 1$ or $i = I$, variables with i subscripts out of range are determined from the periodicity conditions, e.g., $u_{0,j} = u_{I,j}$ and $u_{I+1,j} = u_{1,j}$.

To determine balanced initial conditions for the unscaled version of system (6.1)–(6.3), we assume that the given initial data are the nondivergent wind components u^0 and v^0 and the corresponding vorticity ζ. Then we compute the initial divergence and geopotential from the iteration scheme

$$\delta^{(k+1)} = [u^{(k)}\Phi_x + v^{(k)}\Phi_y]/(\phi_0 - \Phi), \tag{6.8}$$

$$\begin{aligned} \nabla^2\phi^{(k+1)} - [\Phi_x\phi_x^{(k+1)} + \Phi_y\phi_y^{(k+1)}]/\phi_0 = \beta y\zeta - \beta u^{(k)} \\ + 2[u_x^{(k)}v_y^{(k)} - u_y^{(k)}v_x^{(k)}] - u^0\delta_x^{(k)} - v^0\delta_y^{(k)} \\ + [(u^0u_x^0 + v^0u_y^0 - \beta y v^0)\Phi_x + (u^0v_x^0 \\ + v^0v_y^0 + \beta y u^0)\Phi_y]/\phi_0, \end{aligned} \tag{6.9}$$

where $k = 0, \ldots, K$. The boundary conditions for the elliptic equation for ϕ are taken from the limiting form of the equation for v as y approaches $-Y$ or Y, i.e. $\phi_y^{(k+1)} = -\beta y u^{(k)}$. Since this is a singular problem, the right-hand side (rhs) must satisfy a certain constraint in order for a solution to exist. For the simple case when the mountain height is a function of x only, i.e., $\Phi = \Phi(x)$, the constraint is

$$\int_R \exp[-\Phi(x)/\phi_0](\text{rhs})dxdy = \int_0^X \exp[-\Phi(x)/\phi_0]\phi_y\Big|_{-Y}^{Y}dx,$$

where, for the sake of simplicity, we have assumed $\Phi(0) = 0$. If the constraint is satisfied, then ϕ is uniquely determined up to a constant which we choose so that the mean of ϕ is ϕ_0. At any iteration, we can calculate the wind components $u^{(k)}$ and $v^{(k)}$ from ζ and $\delta^{(k)}$ from the relations

$$\nabla^2 u^{(k)} = \delta_x^{(k)} - \zeta_y, \tag{6.10}$$

$$\nabla^2 v^{(k)} = \delta_y^{(k)} + \zeta_x. \tag{6.11}$$

The boundary conditions at $y = -Y$ and $y = Y$ for the elliptic equation for $v^{(k)}$ are naturally chosen to be $v^{(k)} = 0$. The boundary conditions at $y = -Y$ and $y = Y$ for the elliptic equation for $u^{(k)}$ are taken as $u_y^{(k)} = -\zeta$ which determines $u^{(k)}$ uniquely except for a constant which is chosen so that the mean of $u^{(k)}$ is the same as the mean of u^0. The elliptic equations are solved by the direct methods of Swarztrauber and Sweet (1975).

For the numerical results which follow, we obtained the initial nondivergent wind components from the streamfunction ψ given by

$$\psi = -u_0 y - u_1 Y \sin k_1 x(1 - \cos k_2 y)/\pi, \tag{6.12}$$

where u_0 and u_1 are constants, $k_1 = 2\pi/X$, and $k_2 = \pi/Y$. The corresponding nondivergent wind components are

$$\left.\begin{aligned} u^0 &= -\psi_y = u_0 + u_1 \sin k_1 x \sin k_2 y \\ v^0 &= \psi_x = -2u_1 Y \cos k_1 x(1 - \cos k_2 y)/X \end{aligned}\right\}. \tag{6.13}$$

The parameters of the numerical model have the following values:

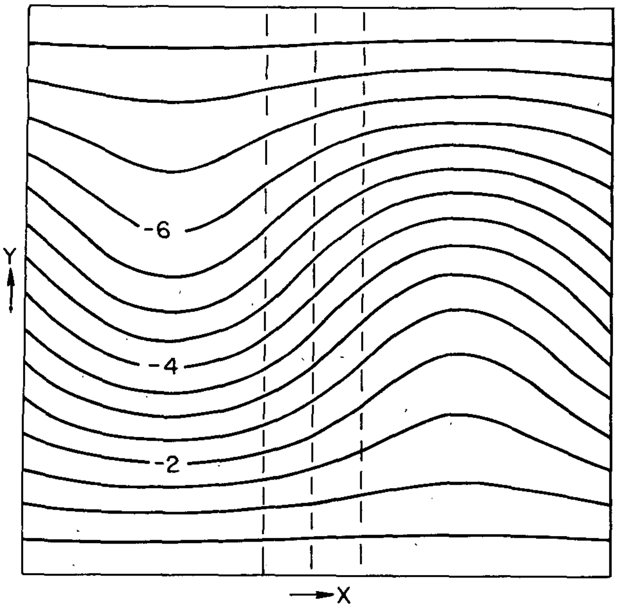

FIG. 1. Solid lines are the initial streamfunction contours in units of 10^8 m^2 s^{-1}. The center dashed line is the mountain ridge and has the value 10^3 m. The dashed lines on either side of the center one have the value 5×10^2 m.

$X = 4000$ km	$u_1 = 10$ m s^{-1}
$Y = 4000$ km	$\phi_0 = 10^5$ m^2 s^{-2}
$I = 120$	$\beta = 2\Omega \cos(0)/r$
$J = 120$	$\Omega = 7.292 \times 10^{-5}$ s^{-1}
$u_0 = 20$ m s^{-1}	$r = 6.37122 \times 10^6$ m.

The orography is assumed to be a function of x only and is given by

$$\Phi = 0.1\phi_0[.5(1 - \cos k_1 x)]^{10}. \quad (6.14)$$

Fig. 1 is a composite contour map of the orography and initial streamfunction. The center dashed line is the mountain ridge and has the value 10^3 m. The dashed line on either side of the center one shows the contour at which the mountain height is half that of the ridge line. The solid lines are the stream function contours in units of 10^8 m^2 s^{-1}. We have deliberately selected a westerly zonal flow in the tropics to test the initialization scheme, because a westerly flow induces a greater global response to the mountain than an easterly flow (Kasahara, 1966).

Figs. 2a–2c show plots of the geopotential deviation ϕ in units of 10^2 m^2 s^{-2} versus time t in hours for the two grid points A with $(i,j) = (31, 61)$ and B with $(i,j) = (91, 76)$ for three different initialization schemes. Fig. 2a shows model results with the initialization scheme where the wind field is assumed nondivergent ($\delta = 0$) and the geopotential is calculated from the balance equation

$$\nabla^2\phi = f\zeta - \beta u^0 + 2(u_x^0 v_y^0 - u_y^0 v_x^0). \quad (6.15)$$

Fig. 2b shows the results of calculations based on the initialization scheme in which the geopotential is calculated from (6.15) and the wind components from (5.12) with the divergence calculated from

$$\delta = (u^0\Phi_x + v^0\Phi_y)/\phi_0. \quad (6.16)$$

It is clear from Fig. 2b that the large amplitude gravity waves shown in Fig. 2a are considerably suppressed by the initialization scheme suitable for the midlatitude beta plane case. However, the short-period (~2 h) oscillations are still visible in Fig. 2b. Fig. 2c shows the results based on the initialization scheme in which the divergence is calculated from (6.8) and the geopotential from (6.9) with $K = 3$ for the case of $\Phi = \Phi(x)$. We see that the last initialization scheme based on bounding the second-order time derivatives almost completely suppresses the short-period oscillations.

7. Conclusions

We have introduced a new approach, called the bounded derivative method, to the data initialization of hyperbolic systems with more than one time scale. The principal steps used to determine adjusted initial conditions are 1) to define the characteristic space and time scales of the motion of interest, 2) to perform a scale analysis of the time-dependent system under consideration to identify the terms that can contribute to large time derivatives, and 3) to constrain those terms to ensure that the time derivatives are of the order of the slow time scale taking into account the boundary conditions which are to be used in the model. The larger the number of time derivatives which are ensured to be of the order of the slow time scale, the smoother the solution will be in time.

We applied the bounded derivative method to the nonlinear shallow-water equations including the effect of orography. Both the midlatitude and equatorial beta plane cases were considered. In the midlatitude case, the quasi-geostrophic initialization scheme derived by Phillips (1960) was obtained as a special case of bounding second-order time derivatives. In the equatorial case, the method produced an initialization scheme which is superior to the scheme considered adequate for the midlatitude case, although more complicated. Tribbia (1979) has applied the Baer-Tribbia (BT) initialization method (see the Introduction) to the latter case. As Leith (1980) and Ballish (1979) have indicated, the BT method is derivable from the application of the bounded derivative method to the system of ordinary differential equations which arise from a spectral approximation. Since the present initialization scheme is obtained by direct application of the bounded derivative method to the system of partial

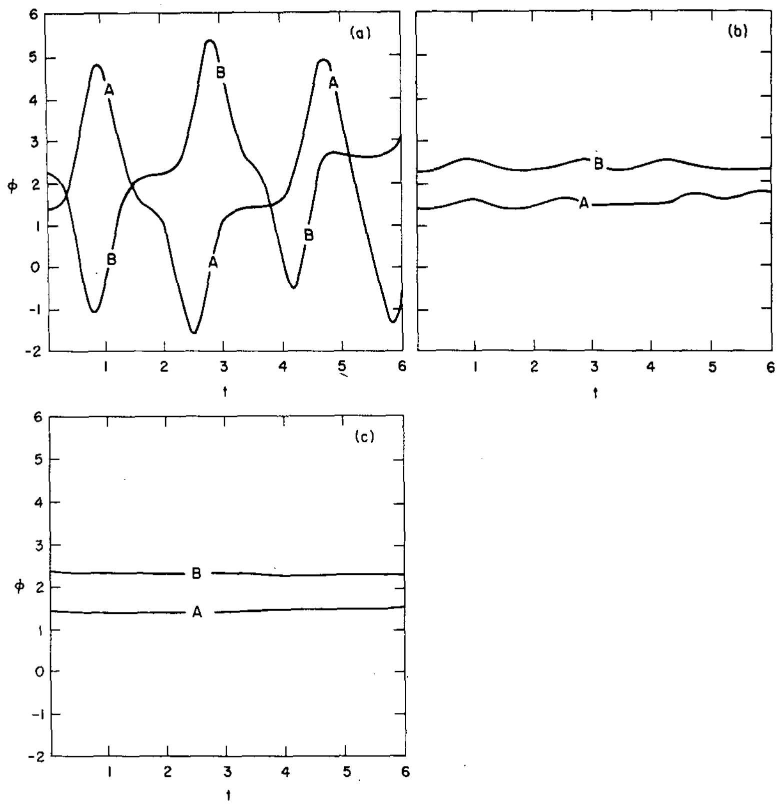

FIG. 2. Figs. 2a–2c show the numerical solutions obtained from initial data with the three initialization schemes discussed in Section 6. Plotted are the geopotential deviation ϕ in units of 10^2 m^2 s^{-2} versus time t in hours for the grid points A with $(i,j) = (31, 61)$ and B with $(i,j) = (91, 76)$.

differential equations, a close relationship exists between the nonlinear normal mode method and the bounded derivative method.

Although we have only demonstrated application of this method to the shallow-water equations, Kreiss (1979, 1980) has discussed its application to more general time-dependent systems of ordinary and partial differential equations. The method can be used for the Cauchy problem where the normal mode technique has proven useful, but it can also be used for initial-boundary value problems where the normal mode technique is not applicable. We have used the method to initialize the shallow-water system in a limited domain with open boundary conditions. This study will be the subject of a separate report.

The bounded derivative method produces constraints that the solution must satisfy if the high-frequency motions are to be kept under control. If these constraints are employed at every time step, the time integration procedure becomes a filtered prediction model in which undesirable motions are eliminated. Browning (1979) used this technique to formulate a nonhydrostatic prediction model through the filtering of sound waves from the Eulerian equation system.

APPENDIX

Initialization of (2.8) When only the Geopotential Deviation ϕ is Given Initially

As we have seen in Section 5, the first-order time derivative is of order unity if and only if (5.1) is satisfied. For the case of the midlatitude beta plane in which the Coriolis parameter f_0 is nonzero, we can solve the first two equations of (5.1) in the form

$$v = f_0^{-1}(\phi_x - \epsilon a), \tag{A1}$$

$$u = f_0^{-1}(-\phi_y - \epsilon b). \tag{A2}$$

We can rewrite constraint (5.2) using (A1) and (A2) as

$$-a_y + b_x = s^{-1}(-\phi_y \Phi_x + \phi_x \Phi_y), \tag{A3}$$

where we have dropped terms of order ϵ. To ensure (5.2) is satisfied at this stage of the initialization, we again neglect terms of order ϵ in (5.1), i.e., we will assume that the right-hand side of (5.1) is identically zero. Since (5.1) contains only two independent equations, we can choose one quantity arbitrarily and for the current case we assume that quantity to be the geopotential deviation ϕ. Then (A1) and (A2) uniquely determine u and v and we see that (5.1) is satisfied at $t = 0$.

As in Section 5, we can find an improved initialization scheme by requiring the second-order time derivatives to be of order unity. But the second-order time derivatives are of order unity if and only if (5.7) is satisfied. We rewrite (5.7) using (A1) and (A2) in the form

$$\begin{bmatrix} 0 & -f_0 & \dfrac{\partial}{\partial x} \\ f_0 & 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} F^{(1)} \\ F^{(2)} \\ F^{(3)} \end{bmatrix}, \tag{A4}$$

where

$$-F^{(1)} = f_0^{-1}(\phi_y\phi_{xx} - \phi_x\phi_{xy}) + \beta y \phi_y,$$

$$-F^{(2)} = f_0^{-1}(\phi_y\phi_{xy} - \phi_x\phi_{yy}) - \beta y \phi_x,$$

$$-F^{(3)} = -\beta y f_0^{-1}\nabla^2\phi - 2f_0^{-2}(\phi_{xx}\phi_{yy} - \phi_{xy}^2) - \beta f_0^{-1}\phi_y,$$

and we have dropped terms of order ϵ. Eq. (A4) constitutes only two independent equations for the three unknowns a, b and c. However, we must also satisfy (A3) which combined with (A4) determines a, b and c uniquely. Since we have dropped terms of order ϵ, the equations for a and b have become an uncoupled system which we can use to determine a and b. Once a and b are determined, u and v can be determined from (A1) and (A2).

Acknowledgments. The authors thank Arne Sundström of the National Defense Research Institute, Sweden, for useful discussions during the formative stage of this work and Norman A. Phillips for his critical review and useful comments on the original manuscript. The authors also benefited from discussions with R. W. Daley, J. Tribbia, and D. L. Williamson. One of the authors, H.-O. Kreiss, acknowledges his support from the National Science Foundation under Grant ATM-76-10218 arranged through New York University.

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